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The sharp Sobolev inequality in quantitative form

Abstract: For $n \geq 2$ and $1 < p < n$, let $p^* := np/(n - p)$. The classical Sobolev inequality states that there exists a positive constant $S(n, p)$ such that

$$S(n, p) \left( \int_{\mathbb{R}^n} |f|^p \right)^{1/p^*} \leq \left( \int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p} , \quad \forall f \in W^{1, p}(\mathbb{R}^n). \quad (1)$$

The largest possible value for $S(n, p)$ is determined by the variational problem

$$S(n, p) = \inf \left\{ \left( \frac{\int_{\mathbb{R}^n} |\nabla f|^p}{\int_{\mathbb{R}^n} |f|^{p^*}} \right)^{1/p^*} : f \in X_p \right\} .$$

The infimum is achieved (i.e., equality holds in (1)) if and only if $f = g_{a, r, x_0}$ ([1], [6], [5]), where

$$g_{a, r, x_0}(x) = \frac{a}{(1 + r|x - x_0|^p)^{(n-p)/p}} , \quad a \neq 0, r > 0, x_0 \in \mathbb{R}^n . \quad (2)$$

Various extensions of this kind have been obtained in the literature, see for instance the paper [3] by H. Brézis and E. Lieb. A natural improvement of (1), conjectured in [3], is an inequality of the form

$$S(n, p)\|f\|_{p^*} (1 + \gamma_p(f)) \leq \|\nabla f\|_p , \quad (3)$$

where $\gamma_p$ is some non-negative function measuring the distance of $f$ from the manifold made up by all the optimal functions (2).

A positive answer to this question for the case $p = 2$ has been proved in [2] by G. Bianchi and H. Egnell. We present a result obtained in [4] which settles the problem for the general case $p \neq 2$. Namely we show that (3) holds with

$$\gamma_p(f) = \left( \inf_{a, r, x_0} \frac{\int_{\mathbb{R}^n} |f - g_{a, r, x_0}|^{p^*}}{\int_{\mathbb{R}^n} |f|^{p^*}} \right)^{\beta(n, p)} ,$$

for every $p \in (1, n)$, and for some exponent $\beta(n, p)$.

References